Effects of contrarians in the minority game

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We study the effects of the presence of contrarians in an agent-based model of competing populations. Contrarians are common in societies. These contrarians are agents who deliberately prefer to hold an opinion that is contrary to the prevailing idea of the commons or normal agents. Contrarians are introduced within the context of the minority game (MG), which is a binary model for an evolving and adaptive population of agents competing for a limited resource. The average success rate among the agents is found to have a nonmonotonic dependence on the fraction a_c of contrarians. For small a_c, the contrarians systematically outperform the normal agents by avoiding the crowd effect and enhance the overall success rate. For high a_c, the antipersistent nature of the MG is disturbed and the few normal agents outperform the contrarians. Qualitative discussion and analytic results for the small a_c and high a_c regimes are presented, and the crossover behavior between the two regimes is discussed.

I. INTRODUCTION

In real-life societies, there always exist some people, referred to as "contrarians," who deliberately prefer to take on an opinion contradictory to the prevailing thoughts of others. Contrarian investment strategies, for example, have been an active and important subject of studies in finances [1,2]. In addition to financial markets, effects of the existence of contrarians have also been studied recently within the context of the dynamics of opinion formation in social systems [3], within the context of the newly developed area of sociophysics. In the present work, we explore how contrarians may affect the global features in one of the most popular agent-based models in recent year, namely, the minority game (MG) [4]. The MG is a binary version of the type of problems related to the bar-attendance problem proposed by Arthur [5]. It has become the basic model of competing populations with built-in adaptive behavior [6]. On the application side, it has been shown that the MG can be suitably generalized to model financial markets and reproduce the so-called stylized facts observed in real markets [7,8].

In the basic MG, an odd number N of agents decide between two possible choices, say 0 or 1, at each time step. The winners are those belonging to the minority group. The winning outcome can, therefore, be represented by a single digit: 0 or 1, according to the winning option. The most recent m winning outcomes constitute the the only information that is made available to all agents. The agents decide based on this global information. For given m, there are 2^m possible m-bit history bit strings, leading to a strategy space consisting of a total of 2^{2^m} possible strategies. Each strategy gives a prediction of either 1 or 0 for each of the 2^m histories. Initially, each agent picks s strategies at random, with repetitions allowed. The performance of the strategies is recorded by assigning (deducting) one (virtual) point to the strategies that

would have predicted the correct (incorrect) outcome, after the outcome is known in a time step. At each time step, each agent follows the prediction of the momentarily best-performing strategy, i.e., the one with the highest virtual points (VPs) among her s strategies. Therefore, a feedback mechanism is built in by allowing the agents to adapt to past performance, which in turn is related to the actions of the agents themselves, by shifting from one strategy to another.

Despite the simplicity of the model, the MG shows very rich and nontrivial properties [9–11]. A quantity of interest, for example, is the standard deviation σ in the number of agents making a particular decision, averaged over different runs. This quantity characterizes the collective efficiency of the system in that a small σ implies a higher success rate or winning probability per agent and hence more winners per turn. Most noticeably, σ exhibits a nonmonotonic behavior on m [9,12,13], showing a minimum at which the performance of the system is better than that of a system in which the agents decide randomly. This feature can readily be explained in terms of the crowd-anticrowd theory of Johnson and co-workers [14,15]. If the number of strategies in the whole strategy pool is smaller than the total number of strategies in play, many agents will hold identical strategies. With decisions based on the best-performing strategies, many agents will then make identical decisions and form a "crowd." For small m, σ is large due to the lack of a cancellation effect from a corresponding "anticrowd" of agents using the opposite or anticorrelated strategy. This small m regime is referred to as the informationally efficient phase, as there is no information in the history bit strings that the agents can exploit [9]. In the large m limit, however, the strategy pool is much larger than the number of strategies in play. Thus it is unlikely that a strategy is being used by more than one agent, and the best performing strategies are those not in play among the agents. In this case, the agents behave as if they are deciding independently and randomly, leading to $\sigma \sim \sqrt{N}/2$. The minimum value of σ occurs at around $2 \times 2^m \sim Ns$, where the size of the crowd and anticrowd become comparable [14,15]. It should be pointed out that the

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machinery in statistical physics of disordered spin systems, most noticeably the replica track [16], has been applied to the large-m regime of the MG [10,17,18]. The approach, which requires random occurrence of the outcomes, is reliable only in the informationally inefficient phase, i.e., from the large-m regime towards where σ exhibits its minimum as m is decreased from above. This restriction on the application of the approach is related to the *relevance* of memory, and hence history, in the efficient phase of the MG [19–23] rather than irrelevance [24]. The effects of the relevance of memory in the efficient phase are known to be particularly significant in heterogeneous systems such as those with a mixed population in which the agents either have different capabilities [20] or follow different sets of strategies in response to a given history [22]. The model to be studied in the present work represents another situation of the latter scenario.

An important and interesting question is whether the collective efficiency in MG can be optimized and how, especially in the efficient regime of the system. A few modified versions of the MG have been proposed and studied, with an enhanced performance to different extent [25–28]. In the present work, we show that a population consisting of a small fraction of agents with contrarian character, i.e., agents who act opposite to the prediction of their own best-performing strategies, will have the standard deviation highly suppressed and hence the performance of the population greatly improved. The plan of the paper is as follows. In Sec. II, we present our model and define the action of the contrarians. In Sec. III, we present results of extensive numerical simulations, together with qualitative discussion and analytic results on the behavior in the limits of small and large fractions of contrarians. The crossover behavior between the two limits is also discussed. Section IV summarizes the present work.

II. MODEL: MINORITY GAME WITH CONTRARIANS

We consider a system of an odd number N of agents, including N_n *normal* agents and N_c *contrarians*, competing for a limited resource. At each time step, each agent must choose one of two options 0 or 1. The winners are those belonging to the minority group and a winner is awarded one (real) point for each winning action. The only information available to all agents is the history bit strings recording the most recent m winning outcomes (i.e., the minority sides). For a given value of m, there are 2^m possible histories. A strategy gives a prediction for each of the 2^m histories, and therefore the whole strategy pool has 2^{2^m} strategies in total. Initially, each agent randomly picks s strategies from the strategy pool, with repetitions allowed. After each time step, each strategy is assessed for its performance on its prediction. For a correct (incorrect) prediction, one (virtual) point (VP) is added (deducted), i.e., the VP of the ith strategy is updated by $\mathrm{VP}_i(t+1) = \mathrm{VP}_i(t) \pm 1$ for correct and incorrect predictions of the winning option at the time step t. The VPs thus reflect the cumulative performance of the strategies that an agent holds from the beginning of a run. At each time step, each agent makes use of the momentarily best-performing strategy, i.e., the one with the highest VP, in her procession for decision. A random tie-breaking rule is used in case of tied VPs. However, normal agents and contrarians use the best-performing strategies in *different* ways. For a normal agent, she *follows* the predictions of the best-performing strategy. For a contrarian, however, she takes the opposite (hence the name contrarian) action to the prediction of her best-performing strategies, i.e., if the strategy with the highest VP says 0, for example, a contrarian will choose option 1. Note that the assignment of VPs to strategies does *not* depend on the type of agents under consideration. For a contrarian, for example, if she loses in a time step, her best-performing strategy has actually predicted the correct outcome and hence a VP will be rewarded. The term "contrarians" is used here in a broad sense, and is not restricted to the meaning within the context of economics and finance. Note that both the normal agents *and* the contrarians aim at winning by predicting the minority group—only their actions after inspecting the best-performing strategy that they hold are different. The aim here is to investigate how the existence of a group of contrarians in the system may affect the performance of the population as a whole as well as the performance of the separate groups of normal agents and contrarians. In the context of market trading, there always exist some groups of agents who interpret economics indicators differently and act differently, although every one may be analyzing the indicators using similar methods or strategies.

The normal agents in the present model can be regarded as the fundamentalists in a related variation on the MG known as the mixed majority-minority game [29]. The fundamentalists are similar to the chartists or technical analysts, who follow the cumulative performance of the strategies. The term contrarians in our model refer to those agents who do not follow the rules in the basic MG. In the case of a small fraction N_c/N of these agents, they decide in the opposite way as most agents do. However, it should be noted that the contrarians are different from the so-called trend followers in the mixed majority-minority game [29]. In Ref. [29], the trend-followers access the strategies' performance on predicting the majority option and use the strategy that is most successful in predicting the majority option for decision. While they behave quite similar to the contrarians in the present model, there is a subtle difference. For example, consider a contrarian who holds the strategy R but not its anticorrelated partner $\bar{R}$ in her initial pick of strategies. A pair of anticorrelated strategies give different predictions for every possible history bit string. At a moment in the game, let R be the best performing among the s strategies she holds. Her action, being a contrarian, will correspond to that of the strategy $\bar{R}$. For a trend follower in Ref. [29] holding the same set of s strategies, she will follow the prediction of one of her $(s-1)$ strategies in addition to R, as R has predicted the minority option quite successfully and hence bas been unsuccessful in predicting the majority option. Her action, being a trend follower, is in general *different* from that of $\bar{R}$, a strategy that she does not hold. As we shall discuss, it is interesting to see that several different variations on the MG [29–31] give qualitatively similar results, as they all amount to allowing some of the agents to break away from the scenario of crowd formation [14,15] in the basic MG.
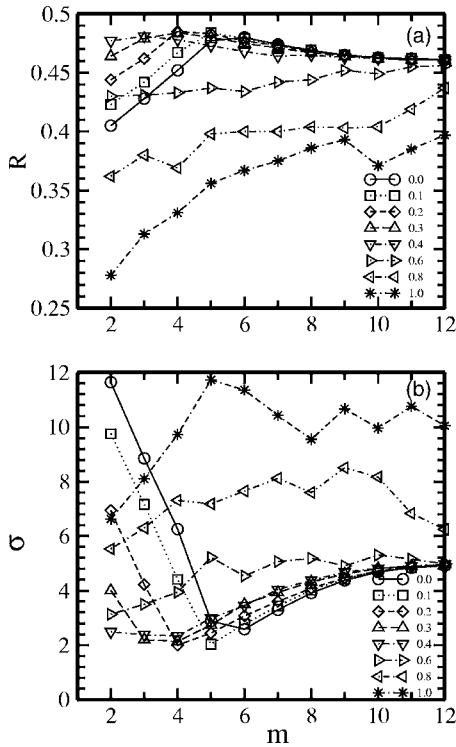
FIG. 1. (a) The averaged success rate R and (b) the averaged standard deviation σ, as a function of the history bit length m based on which agents decide. Numerical results for different fractions of contrarians $a_c = 0.0, 0.1, 0.2, 0.3, 0.4, 0.6, 0.8, 1.0$ are shown in different symbols.

III. RESULTS AND DISCUSSIONS

We have performed extensive numerical simulations to study the effects of the presence of contrarians in MG. Typically, we consider systems of $N=101$ agents, with $s=2$ strategies per agent. Each run lasts for 10^4 time steps and each data point represents an average over the results of 50 independent runs of different initial distributions of strategies and initial histories in starting the runs. Figure 1(a) shows the averaged success rate R over all the agents, which is the number of real points per agent per turn, as a function of the parameter m, for different fractions of contrarians $a_c = N_c/N$ in the system. For $a_c=0$, our model reduces to the MG. The results show that, for small m corresponding to the efficient phase of the basic MG, and for small a_c, R increases as a_c increases. For about $a_c = 0.2 - 0.3$, R achieves a maximum value of about 0.485 at $m=4$. Note that by definition the highest value of R is $(N-1)/2N$. The effects of contrarians are also reflected in the averaged standard deviation σ [see Fig. 1(b)], which drops sensitively with a_c for small values of a_c in the efficient phase. In the absence of contrarians, σ shows a minimum at about $m=6$ for the size of system considered [9]. In the presence of contrarians, the minimum in σ now occurs at $m=4$, with a suppressed value of σ when compared with the basic MG. The success rate and σ are related in that a smaller σ implies more winners per turn and hence a larger success rate [20]. The results in Fig. 1, therefore, indicate that the presence of a small fraction of contrarians enhances the collective performance of the system in the efficient phase (small m) of the MG.
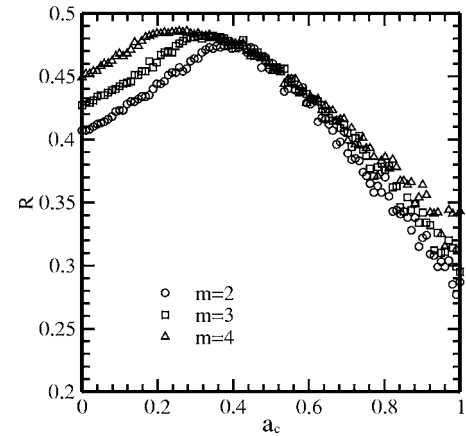

FIG. 2. The average success rate R as a function of the fraction of contrarians a_c, for different values of $m=2$, 3, and 4.

As the key feature of improved performance occurs at small values of m, we will focus in this regime from now on. Figure 2 summarizes the dependence of the averaged success rate R on the fraction of contrarians a_c, for $m=2,3,4$. For small a_c, R increases sensitively with a_c, and achieves a maximum at some value of a_c. Note that both the maximum value of R and the fraction a_c for achieving the maximal R are both m dependent. An overall maximal value of R ≈ 0.485 occurs at $m=4$ and $a_c \approx 0.25-0.3$. For large a_c, R decreases as a_c increases and the corresponding values of R become less sensitive to m. It is also interesting to investigate the success rates averaged over the normal agents and averaged over the contrarians separately, to see how the averaged results in Fig. 2 come about. Figure 3 shows the results for $m=2$, which are typical of the small m cases shown in Fig. 2. The results indicate that a small fraction of contrarians can *systematically* take advantage of the background normal agents, as the contrarians have a success rate that is significantly higher than $1/2$, while the normal agents basically take on a constant success rate corresponding to that of the basic MG with the same value of m. However, as the fraction of contrarians becomes large, it is the remaining few normal
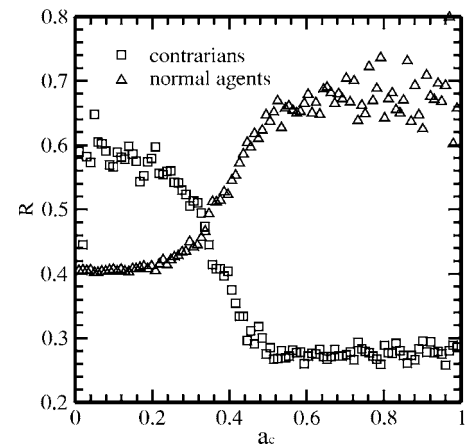

FIG. 3. The averaged success rates R of the normal agents and of the contrarians as a function of the fraction of contrarians a_c. The results are for the case of $m=2$.

agents who take advantage of the contrarians and attain a success rate of about 0.7, while the contrarians only have a success rate of about 0.25. A crossover between these two regimes occurs at an intermediate fraction of $a_c \approx 0.36$ for $m=2$, where the success rates of the two types of agents are comparable.

The behavior for a small fraction of contrarians can be readily understood within the physically transparent crowd-anticrowd picture of the MG [14,15]. For a small fraction of contrarians, the winning outcomes and hence the strategies' VPs are still dominated by the behavior of the normal agents. Therefore, the behavior of the system basically follows that of the basic MG. For the present model, the most important point to realize is the antipersistent nature of the basic MG, i.e., there is no runaway VPs for the strategies [11]. In other words, strategies that have predicted the correct (incorrect) outcomes in recent turns are bound to predict incorrectly (correctly) in future turns. This leads to the so-called doubly periodic behavior in the outcomes [11] as it takes the system about 2×2^m time steps to pass through an Eulerean trial in the history space formed by all the 2^m possible histories [32,33]. For small m, the number of strategies in play is larger than the total number of strategies, implying an appreciable overlap of strategies among the agents. The lower success rate of the normal agents (see Fig. 3) comes about from the crowd effect, i.e., a group of agents using the same or similar better-performing strategies for decision at a time step. For small m, this crowd is too big to win. A low success rate [relative to $(N-1)/2N$] or a large σ implies that there is room for more winners per turn. A contrarian, by taking the opposite action of the prediction of the best-performing strategy that she holds, is given the ability to avoid herself from joining the crowd and win more frequently than the minority rule allows. This breaking away from the crowd has the effect of allowing more winners per turn and hence suppressing the standard deviation [see Fig. 1(b)]. It is worth noting that several variations of the MG also give an enhanced success rate under some condition. For example, the presence of a fraction of agents who decide based on a larger value of m in a background of agents using a smaller value of m also gives rise to an enhanced overall success rate [20]. Another way of breaking away from the crowd is to allow some agents to opt out of a MG at random time steps [30]. Quantitatively, one expects that in the large N limit, since the normal agents have a success rate $R=R(a_c=0) \equiv R(0)$ and the outcomes are dominated by the normal agents for small a_c, the contrarians have a success rate of $1-R(0)$. Averaging over the normal and contrarian agents gives a success rate $R(a_c)=R(0)+a_c$, which is a good approximation of the numerical results in Fig. 2 for small a_c. It is interesting to note that similar features as shown in Fig. 1 have also been observed in different mechanism in avoiding the formation of crowds [31].

As the fraction of contrarians increases, the features in the winning outcome series start to deviate from that of the basic MG (i.e., without the contrarians). In particular, the antipersistence of the strategies' VPs may be destroyed. This will lead to some strategies with runaway VPs, i.e., VPs that keep on increasing or decreasing in a run. Accompanying this effect is the emergence of biased conditional probabilities in
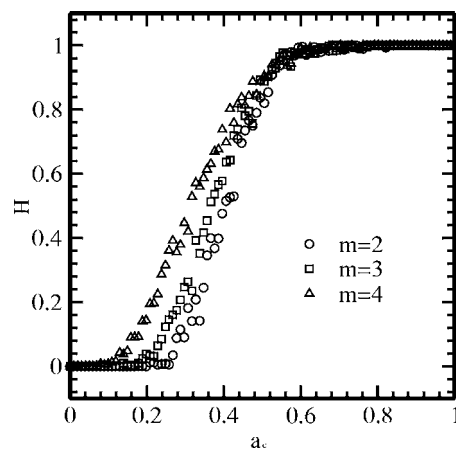

FIG. 4. The quantity H as a function of the fraction of contrarians a_c for $m=2$, 3, and 4. For a range of small a_c, $H=0$ showing that the outcome winning series is dominated by the normal agents. For high a_c, $H=1$ indicating that the antipersistent nature of the system has been disturbed by the presence of contrarians.

the winning outcome series. In the basic MG, the probability of having a winning outcome of 1 following a given history of k bits ($k \leqslant m$) is equal to that of having a winning outcome of 0 following the same k-bit history [9]. To check the change in this basic feature of the MG as the fraction of contrarians increases, we study the quantity [7]

$$H = \frac{1}{Q} \sum_\mu [P(0|\mu) - P(1|\mu)]^2, \qquad (1)$$

where the sum is over all the Q possible histories μ of a certain bit length and $P(i|\mu)$ is the conditional probability of having a winning outcome of i ($i=0$, or 1) given the history bit string is μ. For the basic MG, $H=0$ as the two conditional probabilities cancel. We therefore expect that as a_c increases, $H=0$ in a range of a_c for which the contrarians are too few to affect the outcomes but can efficiently avoid the crowd effect. For large a_c, $H>0$ as the system becomes increasingly deviated from the antipersistent behavior. Figure 4 shows the dependence of H for histories of m bits on the fraction of contrarians, for systems with $m=2,3,4$ Interestingly, the range of a_c with $H=0$ corresponds to the same range that the success rate of the normal agents is flat. The results indicate that the winning outcomes series has similar features as in the basic MG in this range of small a_c, hence justifying our previous discussion on the small a_c behavior. As a_c further increases, H starts to deviate from zero at a m-dependent value of a_c. It indicates that the contrarians are not only simply adapting to the actions of the normal agents, but also affecting the outcomes themselves.

For sufficiently high a_c, $H=1$, indicating a highly biased conditional probability. Numerically we have checked that in many runs at high a_c ($a_c>0.6$), the system shows a persistent outcome (of 1's or 0's). For these runs, it is expected $(1-1/2^s)N=3N/4$ agents *hold* a strategy that predicts the persistent winning outcome regardless of the value of m in the large N limit. It is because the system is now restricted to a tiny portion of the history space and it is the prediction

based only on *one* particular history bit string (out of 2^m) in a strategy that really matters [33]. Due to the minority rule, an outcome series of persistent winning option is not allowed in the absence of contrarians. With a large fraction of contrarians in the $3N/4$ agents holding strategies with runaway VPs, however, they act opposite to the prediction of the strategies and leave the normal agents as winners. Among the few $N(1-a_c)$ normal agents, $3/4$ of them win persistently and $1/4$ of them lose persistently. This gives rise to the high averaged success rate (nearly 0.7) for the normal agents at high a_c (see Fig. 3). For the Na_c contrarians, $1/4$ of them persistently take the winning action as they hold a strategy that persistently predicts a wrong outcome. This gives rise to the averaged success rate of about 0.25 for the contrarians at high a_c, as shown in Fig. 3. If we consider the number of winners collectively, there are $(3N/4)(1-a_c)$ winners from the normal agents and $(N/4)a_c$ winners from the contrarians per turn. This leads to an overall success rate of $R(a_c)$ $=(3/4-a_c/2)$ for sufficiently high a_c. From Figs. 3 and 4, we notice that $a_c > 0.6$ corresponds to the high a_c regime. Substituting $a_c > 0.6$ gives a continuous drop of R as a_c increases towards unity, as observed in Fig. 2. For $a_c=1$, $R=1/4$ for runs with persistent winning outcomes. Note that our argument does not depend on the value of m, as long as the number of agents is sufficiently large. In Fig. 2, we observed that the results for $m=2,3,4$ become less sensitive to m in the high a_c regime, as predicted. The $m=2$ results follow our prediction reasonably well. The discrepancies from the prediction in the $m=3$ and 4 results come from the small size of the system ($N=101$) that we used in the numerical simulations, and the fact that there are runs for which the outcome series is different from a persistent winning option. For example, it is possible to have a series consisting of alternating winning options in the high a_c regime. However, the discussion based on outcomes with persistent winning option does capture the essential underlying physics embedded in the numerical results. The features observed in the small a_c and large a_c regimes are similar to those in the mixed majority-minority game [29], despite of the subtle difference in the models. It is also worth noting that the results are reminiscent of the different phases observed in another model with agents of contrarian attitude and a minority convincing rule [3].

In Fig. 2, $R(a_c)$ shows a peak at a m-dependent crossover value $\bar{a}_c$. Following our discussions in the small a_c and high a_c regimes, we may estimate $\bar{a}_c$ by approximating it as the value of a_c at which the small a_c behavior crosses over to the high a_c behavior, without considering the details of the intermediate regime. Thus, $\bar{a}_c$ can be determined by $R(0)+\bar{a}_c$ $=3/4-\bar{a}_c/2$, giving $\bar{a}_c=1/2-2R(0)/3$. Here $R(0)$ is m dependent due to the better crowd-anticrowd cancellation effect

as m increases. The value $R(0)$ can be obtained by invoking the analytic expressions within the crowd-anticrowd theory [14,15]. For our purpose, it is sufficient to take the value of $R(0)$ from the numerical results at $a_c=0$. Substituting the numerical results (see Fig. 2) of $R(0)=0.405$, 0.425, and 0.45 for $m=2,3,4$, respectively, we obtained $\bar{a}_c=0.23$, 0.217, and 0.20, for $m=2,3,4$. While these values are slightly lower than that shown for the peaks in Fig. 2, the trend of a decreasing $\bar{a}_c$ with m is in good agreement with numerical results. The major source of discrepancy comes from the behavior in the high a_c limit where not all the runs have persistent winning options, as assumed in our argument.

IV. SUMMARY

We proposed and studied a generalized minority game consisting of a fraction of contrarians. These contrarians prefer to hold an opposite opinion to the commons or the normal agents. Within the context of MG, the contrarians are assumed to always take the opposite action as predicted by their momentarily best-performing strategy. For a small fraction of contrarians, the winning outcomes are dominated by the normal agents and the contrarians can systematically outperform the normal agents by avoiding the crowd effect and hence the losing turns of the normal agents. This leads to an enhanced overall success rate of the system at small a_c. However, a larger fraction of contrarians will alter the features in the outcome winning series, as compared to the basic MG. The results indicate that at high fraction of contrarians, the few normal agents have a substantively higher success rate than the contrarians. This is related to the change from antipersistent to runaway behavior in strategy performance, as a_c increases. This change in character leads to a nonmonotonic dependence of the average success rate among all agents as a function of a_c. The small a_c behavior can be understood within the crowd effect in MG and the high a_c behavior is dominated by the runs with persistent winning outcomes. Analytic expressions were given for both the small a_c and high a_c regimes. The dependence of the crossover contrarian fraction $\bar{a}_c$ on m was then estimated from the behavior in the two limits. The trend of a decreasing $\bar{a}_c$ with m was found to be in agreement with results from numerical simulations.

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